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OPTIMAL FEEDBACK STABILIZATION POLICY WITH ASYMMETRIC LOSS FUNCTIONS†

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Abstract—The major emphasis in this paper is to utilize system-theoretic concepts to adapt econometric models via state-space models to solve for optimal stabilization policies involving asymmetric loss functions. Computational feasibility and ease of implementation are regarded as matters of great importance and hence the dimensionality of the problem is given due attention. It is well established that the optimal control for such a problem is in the form of linear state feedback, which provides us with a more natural solution form. An implementable algorithm for this purpose is presented. We also investigate the importance of the time horizon over which optimization is carried out. The analysis is carried out for the deterministic model of the economy, but we discuss situations where uncertainty enters into the model in additive form. We freely borrow ideas from the vast control literature, specifically, from the area of optimal control for linear systems with quadratic cost functions.

1. INTRODUCTION

The aim of reducing fluctuations in an economy has always been a prime concern of economists and policy makers. Given the prevalence of fluctuations in economic activity, it becomes a matter of great concern that the policy maker act to stabilize the system with optimum use of resources to achieve the targets in question.

Sound policy making in turn requires answers to several difficult questions that arise simultaneously such as: (a) the realistic objective that represents the policy maker's preferences when evaluating economic performance, (b) handling competing objectives such as full employment and price stability, (c) the judicious choice among various policy instruments, e.g. monetary and fiscal instruments, (d) the time-frame of the model and (e) determining policies to achieve desired objectives in the most optimal way.

The literature on optimal stabilization policy, based on pioneering work by Theil (1956, 1964), Tinbergen (1952) and Chow (1972, 1973), has addressed the above issues. Various other researchers, for example Pindyck (1973), have indicated the importance of the problem and suggested techniques based on numerous macroeconomic models. The quadratic loss function has been used in all these approaches. This has been the subject of discontent as it implies that underachieving and overachieving a desired target is to be penalized equally, which may not realistically represent the policy maker's preferences when evaluating performances. Hence, incorporating asymmetric preference functions into the analysis appears necessary.

The method proposed by Theil (1956, 1964) does not handle asymmetric loss functions and is also not suitable for imposing terminal conditions. Terminal conditions reflect the penalty attributed through the loss function on not achieving desired targets by the end of the time horizon of the stabilization policy. Additionally, the stabilizing instrument settings obtained by his method depend upon quantities that do not have economic interpretations and are not dynamic in nature. Finally, the computational burden is excessive due to the large dimension of the matrices involved in the solution procedure.

Chow (1973) suggests a dynamic programming approach to solve the stabilization problem, but this method leads to cumbersome results which are again not suitable for asymmetric loss functions. This solution procedure is suitable for his particular state-space model which can be of fairly high dimension.

Friedman (1972, 1975) suggests incorporating asymmetries‡ into the system by using "piecewise"

†Helpful feedback on this paper was obtained from Edward Greenberg at the Department of Economics and Hiro Mukai at the Department of Systems Science and Mathematics at Washington University, St Louis, Mo.
‡This aspect is also addressed by Gupta *et al.* (1975), when studying aggregate stabilization policy under price controls.

quadratic loss functions in Theil's approach; penalties are assigned based on the region in which the target variables lie. He also addresses the problem of deciding the time horizon of stabilization policy by treating it as an endogenous variable. However, the remaining limitations of Theil's approach carry over to his solution procedure as well. Furthermore, Theil's certainty equivalence principle, upon which his analysis is based, does not hold for piecewise quadratic loss functions.†

In this paper, we attempt to answer traditional issues raised in optimal stabilization policy based on system-theoretic concepts. The tools of optimal control have been available for quite some time, and our emphasis is to utilize them in such a way that they match the need of the policy maker and are computationally tractable. The general control theory literature is mired with generalities and it is our aim to adapt these ideas to a realistic economic environment.

We utilize techniques of optimal control based on a state-space formulation of the problem. A minimum realization state-space model of the econometric model is employed to generate "feedback"‡ stabilization rules for instrument settings. Such a solution is attractive as it is more natural that decisions taken at a particular time period be based on information available till that time only. The asymmetries are incorporated by using piecewise quadratic loss functions in a multi-period time horizon framework. Furthermore, an implementable algorithm to construct the optimal instrument settings is presented. Issues relating to the appropriate time horizon for the stabilization policy and uncertainty are also addressed.

Section 2 deals with formulating the econometric model as a state-space model and solving the problem of optimal instrument settings for a quadratic loss function. In the control literature this is referred to as the "tracking problem", and the solution results in two matrix Ricatti difference equations: the "regulator" equation and the "tracking" equation. Section 3 introduces the piecewise quadratic loss criteria to handle asymmetries in the policy maker's objectives. In Section 4, we present an implementable algorithm to solve the asymmetric problem. In Section 5 we discuss the impact of choice of time horizon on the stabilization policies. Finally in Section 6, applications of this technique to stochastic models of the economy are discussed.

2. STATE-SPACE MODEL AND THE TRACKING PROBLEM

2.1. Model

Consider the following deterministic linear econometric model with r lags in its reduced form:

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_r y_{t-r} + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_r x_{t-r},$$

where y_t is a $(p \times 1)$ vector of endogenous variables and x_t is an $(m \times 1)$ vector of the instruments. Parameters α_i and β_i are matrices of suitable dimensions; $\alpha_i \neq 0$ but some of the β_i may be zero. We now transform this econometric model into a state-space model using Aoki's (1976) methodology. The difference equation of r-lags is converted into r equations with single lags.

Let us define the state vector $z_t = [z_t^1, z_t^2, \ldots, z_t^r]^{\mathrm{T}}$ of dimension $(rp \times 1)$ for $t \geqslant 1$ by the following:

$$z_t^1 = y_t,$$

$$z_t^k = \alpha_k z_{t-1}^1 + z_{t-1}^{k+1} + \beta_k x_{t-1}, \quad k = 2, 3, \ldots, r-1,$$

and

$$z_t^r = \alpha_r z_{t-1}^1 + \beta_r x_{t-1}.$$

The initial state at time $t = 0$ can be written as $z_0 = [y_0, 0, \ldots, 0]^{\mathrm{T}}$. Hence we obtain the state space representation as

$$z_{t+1} = Az_t + Bx_t,$$

$$y_t = Cz_t,$$

†This fact is acknowledged by Friedman (1972) in his paper.

‡The importance of feedback control is discussed by Kalchbrenner and Tinsley (1976), where they present an empirical study for this problem.

where

$$A = \begin{bmatrix} \alpha_1 & I_p & 0 & \dots & 0 \\ \alpha_2 & 0 & I_p & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{r-1} & 0 & 0 & & I_p \\ \alpha_r & 0 & 0 & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_1 \end{bmatrix}, \quad C = [I_p \quad 0 \quad \dots \quad 0].$$

Here I_p is a $p \times p$ identity matrix.

Let us define $n = rp$, then the vector z_t is $n \times 1$, x_t is $m \times 1$ and y_t is $p \times 1$. Also, the matrix A is $n \times n$, B is $n \times m$ and C is $p \times n$. This is a minimum dimension state-space representation of the econometric equation.†

2.2. Tracking problem

As seen in the above section the state-space model of the economy is

$$z_{t+1} = Az_t + Bx_t,$$

$$y_t = Cz_t,$$

with the initial condition $y_0 = \mathscr{y}$ given, i.e. $z_0 = [\mathscr{y}, 0, \dots, 0]^{\mathrm{T}}$, since the endogenous variables y_0 at time $t = 0$ are known.

Consider the following symmetric quadratic loss criterion for a finite time horizon T:

$$L = \tfrac{1}{2}(y_T - \tilde{y}_T)^T S(y_T - \tilde{y}_T) + \frac{1}{2}\sum_{t=0}^{T-1} [(y_t - \tilde{y}_t)^T Q_t (y_t - \tilde{y}_t) + (x_t - \tilde{x}_t)^T R_t (x_t - \tilde{x}_t)],$$

where Q_t and R_t are diagonal matrices and $Q_t \geqslant 0$, $R_t > 0$ for all $t = 0, 1, \dots, T-1$ and $S > 0$. The term $\tilde{y}_t$ is the long-run desired path of the endogenous variables which the policy maker wants to "track". Similarly $\tilde{x}_t$ is the long-run desired path of the instruments. The matrix, Q_t, represents the penalty imposed on deviations of the endogenous variables from their desired path $\tilde{y}_t$. Note that some of the diagonal elements of the Q_t matrix may be zero, i.e. the corresponding endogenous variables may not be targetted. Similarly, matrix R_t represents the penalties associated with the instrument variables. Finally the matrix S reflects the "terminal conditions" of the problem. The S matrix plays a prominent role in deciding the time horizon T of the stabilization policy (this aspect is studied in detail in Section 5). In this analysis Q_t and R_t are time varying matrices, which provides the policy maker with greater flexibility.

Our aim is to minimize the loss function L and solve for the optimal instrument settings x_t^*, $t = 0, 1, \dots, T-1$, and the corresponding target variables y_t^* for $t = 1, 2, \dots, T$. The most convenient way to solve this optimization problem is by using the Discrete Minimum principle (see Seirstad and Sydsaeter, 1977) which is the discrete version of Pontryagin's Minimum principle (see Hestenes, 1966).

Prior to applying this principle to our problem, we consider the Discrete Minimum principle in a general setting.

Let

$$z_{t+1} = f(z_t, x_t, t), \quad t = t_0, \dots, t_f - 1,$$

$$x_t \in \mathscr{X},$$

where $\mathscr{X}$ is a given set in $\mathbb{R}^m$ and t_0 and t_f are fixed integers representing the starting and the final times respectively. We want to find an admissible sequence $x_t, t = t_0, \dots, t_f - 1$ in order to minimize the generalized cost function given by

$$J = \Theta(z_{t_f}, t_f) + \sum_{t=t_0}^{t_f-1} \phi(z_t, x_t, t).$$

†See Aoki (1976) for the minimum dimension state-space representation.

Here Θ is the cost associated with the final time t_f and ϕ is the cost that depends upon the system path from time t_0 to $t_f - 1$.

We now state the Discrete Minimum principle† for the above problem. Let x_t^*, $t = t_0, \ldots, t_f - 1$ be an optimal sequence and let z_t^*, $t = t_0, \ldots, t_f$ be the response to x_t^* through the system equations. If for each $t = t_0, \ldots, t_f - 1$; $\Theta : \mathbb{R}^n \to \mathbb{R}$, $\phi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions and the set $\{f(z, x, t) : x \in \mathscr{X}\}$ is convex for all $z \in \mathbb{R}^n$, then there exists a nonzero vector λ_t^* satsifying

$$\lambda_t^* = \frac{\partial H}{\partial z_t^*}(z_t^*, x_t^*, \lambda_{t+1}^*, t), \quad t = t_0, \ldots, t_f - 1, \tag{1}$$

$$\lambda_{t_f}^* = \frac{\partial \Theta(z_{t_f}, t_f)}{\partial z_{t_f}^*} \tag{2}$$

and

$$H(z_t^*, x_t^*, \lambda_{t+1}^*, t) = \min_{x \in \mathscr{X}} H(z_t^*, x_t, \lambda_{t+1}^*, t),$$

where

$$H(z_t, x_t, \lambda_{t+1}, t) = \phi(z_t, x_t, t) + \lambda_{t+1}^T f(z_t, x_t, t).$$

Function H is the Hamiltonian of the optimal control problem. When $\mathscr{X} = \mathbb{R}^m$, i.e. the admissible set of instruments is the m-dimensional Euclidean space, we have

$$\frac{\partial H}{\partial x_t}(z_t^*, x_t^*, \lambda_{t+1}^*, t) = 0, \quad t = t_0, \ldots, t_f - 1. \tag{3}$$

In our problem, the above introduced functions are of the following form:

$$\begin{aligned}
f &= Az_t + Bx_t, \\
\Theta &= \tfrac{1}{2}(y_T - \tilde{y}_T)^T S(y_T - \tilde{y}_T) = \tfrac{1}{2}(Cz_T - \tilde{y}_T)^T S(Cz_T - \tilde{y}_T), \\
\phi &= \tfrac{1}{2}[(y_t - \tilde{y}_t)^T Q_t(y_t - \tilde{y}_t) + (x_t - \tilde{x}_t)^T R_t(x_t - \tilde{x}_t)] \\
&= \tfrac{1}{2}[(Cz_t - \tilde{y}_t)^T Q_t(Cz_t - \tilde{y}_t) + (x_t - \tilde{x}_t)^T R_t(x_t - \tilde{x}_t)],
\end{aligned}$$

where $Q_t \geqslant 0$ and $R_t > 0$ for all $t = 0, 1, 2, \ldots, T - 1$, and $\mathscr{X} = \mathbb{R}^m$.

Clearly for all $t = 0, 1, 2, \ldots, T - 1$, the functions Θ, ϕ and f are continuously differentiable functions in all their arguments. Furthermore $\{f(z, x, t) : x \in \mathscr{X}\}$ is convex for all $z \in \mathbb{R}^n$, since f is a linear operator on $\mathscr{X}$, which is convex (see Royden, 1965, p. 181). Hence, all the assumptions needed for the Discrete Minimum principle are satisfied by our problem formulation and we can go on to apply it as follows:

The Hamiltonian for our problem is

$$H_t = \tfrac{1}{2}(y_t - \tilde{y}_t)^T Q_t(y_t - \tilde{y}_t) + \tfrac{1}{2}(x_t - \tilde{x}_t)^T R_t(x_t - \tilde{x}_t) + \lambda_{t+1}^T(Az_t + Bx_t).$$

Substituting for y_t, we obtain

$$H_t = \tfrac{1}{2}(Cz_t - \tilde{y}_t)^T Q_t(Cz_t - \tilde{y}_t) + \tfrac{1}{2}(x_t - \tilde{x}_t)^T R_t(x_t - \tilde{x}_t) + \lambda_{t+1}^T(Az_t + Bx_t).$$

Suppressing the asterisk‡ symbol for notational simplicity, from equation (1) we get

$$\lambda_t = \frac{\partial H_t}{\partial z_t} = C^T Q_t(Cz_t - \tilde{y}_t) + A^T \lambda_{t+1}. \tag{4}$$

This equation is called the adjoint equation.

From equation (2) we get the terminal condition as

$$\lambda_T = \frac{\partial}{\partial z_T}(Cz_T - \tilde{y}_T)^T S(Cz_T - \tilde{y}_T) = C^T S[Cz_T - \tilde{y}_T]. \tag{5}$$

†This version is drawn from Sage and White (1977, Chapter 6).

‡Asterisk symbol denotes the optimal sequences.

Finally from equation (3) we obtain

$$\frac{\partial H}{\partial x_t} = 0 = R_t(x_t - \tilde{x}_t) + B^T\lambda_{t+1}$$

and hence

$$x_t = -R_t^{-1}B^T\lambda_{t+1} + \tilde{x}_t. \tag{5a}$$

Replacing x_t into the state equation and the adjoint equation (4), we get

$$z_{t+1} = Az_t - BR_t^{-1}B^T\lambda_{t+1} + B\tilde{x}_t \tag{6}$$

and

$$\lambda_t = C^TQ_t[Cz_t - \tilde{y}_T] + A^T\lambda_{t+1}, \tag{7}$$

with the initial condition as

$$z_0 = [y, 0, \dots, 0]^T \tag{6a}$$

and the terminal condition as

$$\lambda_T = C^TS[Cz_T - \tilde{y}_T]. \tag{7a}$$

Equation (6) is a "forward" difference equation and equation (7) is a "backward" difference equation. Both equations are dependent upon each other for the values of λ_t and z_t respectively. Thus, the above equations form a coupled boundary value problem, which is generally difficult to solve. However, if we can guess the form of the solution then this problem can be simplified. For quadratic loss criterion and linear systems it is well known that the costate variables, λ_t, will be affine with respect to z_t. Hence, let the costate variables be of the form

$$\lambda_t = P_tz_t + w_t,$$

where P_t is an $n \times n$ positive semi-definite matrix and w_t is an $n \times 1$ vector. Substituting for λ_t in equations (6) and (7), we obtain

$$z_{t+1} = Az_t - BR_t^{-1}B^TP_{t+1}z_{t+1} - BR_t^{-1}B^Tw_{t+1} + B\tilde{x}_t \tag{8}$$

and

$$P_tz_t = C^TQ_tCz_t - C^TQ_t\tilde{y}_t + A^TP_{t+1}z_{t+1} + A^Tw_{t+1} - w_t. \tag{9}$$

Solving for z_{t+1} from equation (8)

$$z_{t+1} = K_t^{-1}[Az_t - BR_t^{-1}B^Tw_{t+1} + B\tilde{x}_t], \tag{9a}$$

where $K_t = I + BR_t^{-1}B^TP_{t+1}$. Note that K_t is positive definite and hence its inverse exists.

Replacing z_{t+1} from equation (9a) into equation (9) yields

$$P_tz_t = [C^TQ_tC + A^TP_{t+1}K_t^{-1}A]z_t + A^TP_{t+1}K_t^{-1}B\tilde{x}_t - C^TQ_t\tilde{y}_t + [A^T - A^TP_{t+1}K_t^{-1}BR_t^{-1}B^T]w_{t+1} - w_t.$$

Since the above equation must hold for any arbitrary z_t, we can equate the coefficients of z_t on both sides, to get

$$P_t = C^TQ_tC + A^TP_{t+1}K_t^{-1}A \tag{10}$$

and

$$w_t = [A^T - A^TP_{t+1}K_t^{-1}BR_t^{-1}B^T]w_{t+1} + A^TP_{t+1}K_t^{-1}B\tilde{x}_t - C^TQ_t\tilde{y}_t. \tag{11}$$

Furthermore, from the definition of λ_t and equation (7a), we have

$$\lambda_T = P_Tz_T + w_T = C^TSCz_T - C^TS\tilde{y}_T,$$

which implies that

$$P_T = C^TSC$$

and

$$w_T = -C^T S\tilde{y}_T.$$

These are the terminal conditions for the backward difference equations (10) and (11).

We now examine equation (10) further. Replacing K_t by its definition, we obtain

$$P_t = C^T Q_t C + A^T P_{t+1}[I + BR_t^{-1}B^T P_{t+1}]^{-1}A.$$

Note that evaluating this equation would involve inversions of $n \times n$ matrices; n could be a large number since $n = rp$, where r is the number of lags and p is the number of endogenous variables. Hence, if possible the computational burden of inverting $n \times n$ matrices should be avoided. For this purpose the matrix inversion lemma of Sage and White (1977, pp. 405–406) can be employed to transform the above equation such that inversions of only $m \times m$ matrices are involved, where m represents the number of instruments.

Using the matrix inversion lemma, we obtain

$$P_t = C_t Q_t C + A^T P_{t+1} A - A^T P_{t+1} B[B^T P_{t+1} B + R_t]^{-1} B^T P_{t+1} A, \tag{12}$$

where $P_T = C^T SC$.

Now this involves inverting $m \times m$ matrices only which are usually of lower dimension than $n \times n$ matrices. This equation is a matrix Ricatti difference equation. It can be solved backwards from the time T to time 0, since rest of the quantities are known. In application, the values of P_t for $t = 0, 1, \ldots, T$ are computed and stored in computer memory.

Let us now consider equation (11), i.e.

$$w_t = A^T[I - P_{t+1}K_t^{-1}BR_t^{-1}B^T]w_{t+1} + A^T P_{t+1}K_t^{-1}B\tilde{x}_t - C^T Q_t \tilde{y}_t.$$

Replacing K_t by its definition and applying the matrix inversion lemma, we obtain

$$\begin{aligned} w_t = A^T[I - P_{t+1}\{I - B(R_t + B^T P_{t+1}B)^{-1}B^T P_{t+1}\}BR_t^{-1}B^T]w_{t+1} \\ + A^T P_{t+1}\{I - B(R_t + B^T P_{t+1} B)^{-1} B^T P_{t+1}\}B\tilde{x}_t - C^T Q_t \tilde{y}_t, \end{aligned} \tag{13}$$

with $w_T = -C^T S\tilde{y}_T$.

This is also a matrix difference equation and involves $m \times m$ matrix inversions. To solve this equation, we require the values of P_t, which have already been computed from equation (12) and stored in computer memory.

We are now ready to solve for the optimal instrument settings. Recall from equation (5a) that the optimal x_t are given by

$$\begin{aligned} x_t^* &= -R_t^{-1}B^T\lambda_{t+1} + \tilde{x}_t \\ &= -R_t^{-1}B^T[P_{t+1}z_{t+1} + w_{t+1}] + \tilde{x}_t. \end{aligned}$$

Substituting for z_{t+1} from equation (9a), we get

$$\begin{aligned} x_t^* &= -R_t^{-1}B^T[P_{t+1}K_t^{-1}(Az_t - BR_t^{-1}B^T w_{t+1} + B\tilde{x}_t) + w_{t+1}] + \tilde{x}_t \\ &= -R_t^{-1}B^T P_{t+1}K_t^{-1}Az_t + R_t^{-1}B^T P_{t+1}K_t^{-1}[BR_t^{-1}B^T w_{t+1} - B\tilde{x}_t] - R_t^{-1}B^T w_{t+1} + \tilde{x}_t. \end{aligned}$$

Replacing for K_t^{-1} and using the matrix inversion lemma, we obtain

$$\begin{aligned} x_t^* = &-R_t^{-1}B^T P_{t+1}[I - B(R_t + B^T P_{t+1}B)^{-1}B^T P_{t+1}]Az_t \\ &+ R_t^{-1}B^T P_{t+1}[I - B(R_t + B^T P_{t+1}B)^{-1}B^T P_{t+1}] \\ &\times [BR_t^{-1}B^T w_{t+1} - B\tilde{x}_t] - R_t^{-1}B^T w_{t+1} + \tilde{x}_t. \end{aligned} \tag{14}$$

This is the optimal feedback law for the problem at hand. From the state-space equations, we can obtain the optimal target variables, y_t^*, that are generated when the optimal instrument settings, x_t^*, are used, as follows:

$$z_{t+1}^* = Az_t^* + Bx_t^*$$

with $z_0 = [y, 0, \ldots, 0]^T$, which is known and $y_t^* = Cz_t^*$.

3. ASYMMETRIC LOSS CRITERION

As discussed earlier, it is unrealistic to assume that deviations of targets and instruments from their desired values should be penalized equally. Not only does it not reflect the policy maker's true preferences, it also results in improper instrument settings by attributing erroneous trade-offs. We model the asymmetry in the penalty when the targets are underachieved or overachieved, by using Friedman's (1975) piecewise quadratic loss functions.

The quadratic loss function used in Section 2 was

$$L = \tfrac{1}{2}(y_T - \tilde{y}_T)S(y_T - \tilde{y}_T) + \frac{1}{2}\sum_{t=0}^{T-1} [(y_t - \tilde{y}_t)^T Q_t (y_t - \tilde{y}_t) + (x_t - \tilde{x}_t)^T R_t (x_t - \tilde{x}_t)],$$

where R_t was a positive definite diagonal matrix and S and Q_t were diagonal positive semi-definite matrices.

For notational ease and to avoid multiple subscripts and superscripts, in this section we write $x(t)$ for x_t and $y(t)$ for y_t, etc. Let $x_i(t)$, $i = 1, 2, \ldots, m$ be the ith component of the instrument variable and $y_j(t)$, $j = 1, 2, \ldots, p$ be the jth component of the target variable. Then there are corresponding deviations, $\hat{x}_i(t) = x_i(t) - \tilde{x}_i(t)$ and $\hat{y}_i(t) = y_i(t) - \tilde{y}_i(t)$, which result in a loss.

If the ith instrument, $x_i(t)$ is greater than $\tilde{x}_i(t)$, then let the cost associated with the deviation be $\bar{R}_{ii}(t)$ and if $x_i(t)$ is less than $\tilde{x}_i(t)$, then let it be $\underline{R}_{ii}(t)$. Similarly, for the target variables we have the asymmetric costs $\bar{Q}_{jj}(t)$ associated to overachieving and $\underline{Q}_{jj}(t)$ associated to under-achieving.

This situation is depicted diagramatically in Fig. 1.

The values of $\bar{R}_{ii}(t)$ and $\underline{R}_{ii}(t)$ must be positive, since we require that the weighting matrix of the instruments be positive definite. However, $\bar{Q}_{jj}(t)$ and $\underline{Q}_{jj}(t)$ may be nonnegative.

Let $R_{ii}(t)$ be the ith diagonal element of the $R(t)$ matrix. Thus depending upon the region in which the $x_i(t)$ lie, the $R(t)$ matrix takes different values over time:

$$R_{ii}(t) = \begin{cases} \underline{R}_{ii}(t), & x_i(t) < \tilde{x}_i(t), \\ \bar{R}_{ii}(t), & x_i(t) \geqslant \tilde{x}_i(t), \end{cases} \tag{15}$$

for $i = 1, 2, \ldots, m$.

Similarly for the target variables, $y(t)$, let the weighting matrix $Q(t)$ consist of the diagonal elements, $Q_{jj}(t)$ given by

$$Q_{jj}(t) = \begin{cases} \underline{Q}_{jj}(t), & y_j(t) < \tilde{y}_j(t), \\ \bar{Q}_{jj}(t), & y_j(t) \geqslant \tilde{y}_j(t), \end{cases} \tag{16}$$

for $j = 1, 2, \ldots, p$.

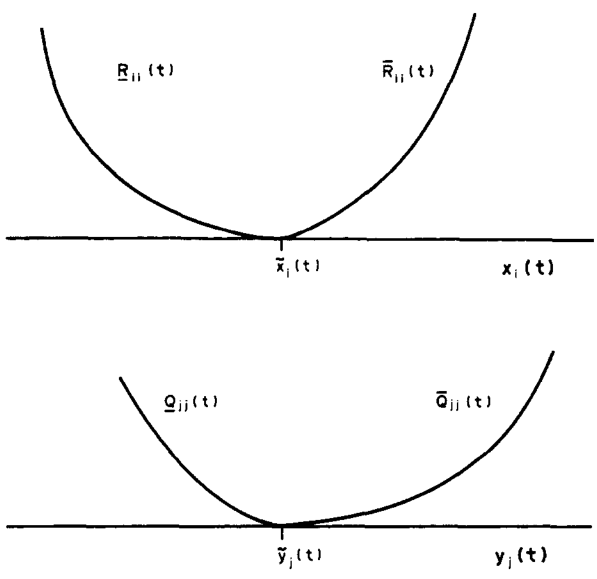

Fig. 1

If some target variable $y_j(t)$ is not being targetted, then the associated $Q_{jj}(t)$ is zero for all time t.

Finally, the terminal weighting matrix S can be treated in an identical manner:

$$S_{jj} = \begin{cases} \underline{S}_{jj}, & y_j(T) < \tilde{y}_j(T), \\ \bar{S}_{jj}, & y_j(T) \geqslant \tilde{y}_j(T), \end{cases} \tag{17}$$

$j = 1, 2, \ldots, p$.

3.1. Discrete minimum principle for the piecewise-quadratic loss function

In this section, we will show that the piecewise quadratic loss function satisfies the assumptions of the Discrete Minimum principle. Using the new definitions of the weighting matrices, our asymmetric cost function becomes

$$\begin{aligned} L = {} & \tfrac{1}{2}\{(y_T - \tilde{y}_T)^+\}^T \bar{S}(y_T - \tilde{y}_T)^+ + \tfrac{1}{2}\{(y_T - \tilde{y}_T)^-\}^T \underline{S}(y_T - \tilde{y}_T)^- \\ & + \frac{1}{2}\sum_{t=0}^{T-1} [\{(y_t - \tilde{y}_t)^+\}^T \bar{Q}_t (y_t - \tilde{y}_t)^+ + \{(y_t - \tilde{y}_t)^-\}^T \underline{Q}_t (y_t - \tilde{y}_t)^- \\ & + \{(x_t - \tilde{x}_t)^+\}^T \bar{R}_t (x_t - \tilde{x}_t)^+ + \{(x_t - \tilde{x}_t)^-\}^T \underline{R}_t (x_t - \tilde{x}_t)^-], \end{aligned} \tag{18}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

In this formulation, the functions introduced during the statement of the Discrete Minimum principle in Section 2.2 take the following form:

$$f = Az_t + Bx_t,$$

$$\Theta = \tfrac{1}{2}\{(y_T - \tilde{y}_T)^+\}^T \bar{S}(y_T - \tilde{y}_T)^+ + \tfrac{1}{2}\{(y_T - \tilde{y}_T)^-\}^T \underline{S}\{y_T - \tilde{y}_T)^-, \tag{19}$$

$$\begin{aligned} \phi = {} & \tfrac{1}{2}[\{(y_t - \tilde{y}_t)^+\}^T \bar{Q}_t (y_t - \tilde{y}_t)^+ + \{(y_t - \tilde{y}_t)^-\}^T \underline{Q}_t (y_t - \tilde{y}_t)^- \\ & + \{(x_t - \tilde{x}_t)^+\}^T \bar{R}_t (x_t - \tilde{x}_t)^+ + \{(x_t - \tilde{x}_t)^-\}^T \underline{R}_t (x_t - \tilde{x}_t)^-]. \end{aligned} \tag{20}$$

Clearly, for all $t = 0, 1, 2, \ldots, T-1$, the functions Θ and ϕ are continuously differentiable in x and y. Furthermore, since the function f is the same as in Section 2.2; $\{f(z, x, t) : x \in \mathscr{X}\}$ is convex for all $z \in \mathbb{R}^n$. Hence, all the assumptions of the Discrete Minimum principle are satisfied.

Now we present an algorithm to solve the stabilization problem with an asymmetric loss function using the results obtained in Section 2 for the quadratic loss function.

4. IMPLEMENTABLE ALGORITHM FOR STABILIZATION PROBLEM WITH ASYMMETRIC LOSS

In this section, we discuss the solution procedure to solve the stabilization problem with asymmetric loss function. The solution algorithm is an iterative algorithm in which the solution of the problem with quadratic loss function is repetitively evaluated. At each iteration, the $Q(t)$, $R(t)$ and S matrices are updated depending upon the region in which the value of the instruments, $x^*(t)$, and the value of the targets, $y^*(t)$, that are generated in that iteration lie. This procedure is continued till no more updating is required. The steps involved are summarized in the following algorithm.

Algorithm

Parameters. Initial state, $z_0 = [\mathscr{y}, 0, \ldots, 0]^T$ and final time, T.

Step 1. Set $Q_{jj}(0)$ depending upon the value of $\mathscr{y}$.

Step 2. Set $Q_{jj}(t) = \bar{Q}_{jj}(t)$, $j = 1, 2, \ldots, p$ and $t = 1, 2, \ldots, T-1$;

$S_{jj} = \bar{S}_{jj}$, $j = 1, 2, \ldots, p$;

$R_{ii}(t) = \bar{R}_{ii}(t)$, $i = 1, 2, \ldots, m$ and $t = 0, 1, 2, \ldots, T-1$.

Step 3. Compute $x^*(t)$, $t = 0, 1, 2, \ldots, T-1$ and $z^*(t)$, $t = 1, 2, \ldots, T$ using equation (14) and the state-space equation: $z^*_{t+1} = Az^*_t + Bx^*_t$. Then find $y^*(t)$, $t = 1, 2, \ldots, T$ using $y^*_t = Cz^*_t$.

Step 4. Depending upon the region in which $x^*(t)$ and $y^*(t)$ lie, update $Q(t)$, $R(t)$ and S matrices, using equations (15)–(17).

Step 5. If there are no changes in $Q(t)$, $R(t)$ and S matrices then STOP; else go to Step 3.

Steps 1 and 2 are the initialization steps. In Step 1, we set the value of the matrix $Q_{jj}(0)$ depending upon the initial values of the endogenous variable y. In Step 2, we set the matrices $Q_{jj}(t)$ for $t = 1, 2, \ldots, T-1$, $R_{ii}(t)$ for $t = 1, 2, \ldots, T$ and matrix S to their upper values. In Step 3, the optimal feedback instrument settings $x^*(t)$ are obtained using equation (14). This involves solving two backward Ricatti matrix difference equations. Once the optimal instrument settings are known, optimal state $z^*(t)$ and the optimal target variables $y^*(t)$ are found. Note that these "optimal" values depend upon the Q, R and S matrices. Now, depending upon the region in which the values of $x^*(t)$ and $y^*(t)$ lie, these matrices are updated, using equations (15)–(17). If there are no changes in these matrices, then the algorithm stops.

4.1. Convergence of the algorithm

Let us consider the case when the algorithm stops at some solution point (x^*, y^*). We want to show that this point satisfies the necessary conditions given by the Discrete Minimum principle.

From equations (18)–(20), the Hamiltonian for the asymmetric loss function is given by

$$H_t = \frac{1}{2}[(y_t - \tilde{y}_t)^+\}^T\bar{Q}_t(y_t - \tilde{y}_t)^+ + \{(y_t - \tilde{y}_t)^-\}^T\underline{Q}_t(y_t - \tilde{y}_t)^- + \{(x_t - \tilde{x}_t)^+\}^T\bar{R}_t(x_t - \tilde{x}_t)^+ + \{(x_t - \tilde{x}_t)^-\}^T\underline{R}_t(x_t - \tilde{x}_t)^-] + \lambda_{t+1}^T(Az_t + Bx_t). \quad (21)$$

Replacing for y_t with $y_t = Cz_t$ and then differentiating with respect to z_t, we get

$$\lambda_t = \frac{\partial H_t}{\partial z_t} = C^T\bar{Q}_t(Cz_t - \tilde{y}_t)^+ - C^T\underline{Q}_t(Cz_t - \tilde{y}_t)^- + A^T\lambda_{t+1} \quad (22)$$

and similarly differentiating with respect to x_t, we obtain

$$\frac{\partial H}{\partial x_t} = \bar{R}_t(x_t - \tilde{x}_t)^+ - \underline{R}_t(x_t - \tilde{x}_t)^- + B^T\lambda_{t+1}. \quad (23)$$

Substituting with $y_t = y_t^* = Cz_t^*$ and $x_t = x_t^*$ in equations (22) and (23), we obtain

$$\lambda_t = \left.\frac{\partial H_t}{\partial z_t}\right|_{z_t = z_t^*} = C^T\bar{Q}_t(Cz_t^* - \tilde{y}_t)^+ - C^T\underline{Q}_t(Cz_t^* - \tilde{y}_t)^- + A^T\lambda_{t+1},$$

$$\left.\frac{\partial H}{\partial x_t}\right|_{x_t = x_t^*} = \bar{R}_t(x_t^* - \tilde{x}_t)^+ - \underline{R}_t(x_t^* - \tilde{x}_t)^- + B^T\lambda_{t+1}.$$

Now using the definitions of Q_t and R_t from equations (16) and (15), we get

$$\lambda_t = \left.\frac{\partial H_t}{\partial z_t}\right|_{z_t = z_t^*} = C^TQ_t(Cz_t^* - \tilde{y}_t) + A^T\lambda_{t+1},$$

$$\left.\frac{\partial H}{\partial x_t}\right|_{x_t = x_t^*} = R_t(x_t^* - \tilde{x}_t) + B^T\lambda_{t+1}.$$

These equations are same as equations (4) and (5a) of Section 2.2 and will lead to equation (14), which is satisfied by (x^*, y^*). Hence, (x^*, y^*) would obviously satisfy equations (4) and (5a) and thus the necessary conditions of the Discrete Minimum principle.

In case the algorithm does not stop then it "jams" at some point (x^*, y^*), i.e. the algorithm cycles back to the same point (x^*, y^*) after a finite number of iterations. This can happen if the magnitude of the penalties (Q_{jj}, S_{jj} and R_{ii}) are extremely high for a large number of target and instrument variables. This reflects the need to strictly adhere to the "desired path" of the target/instrument variables; which is not frequent in most economic policy applications.

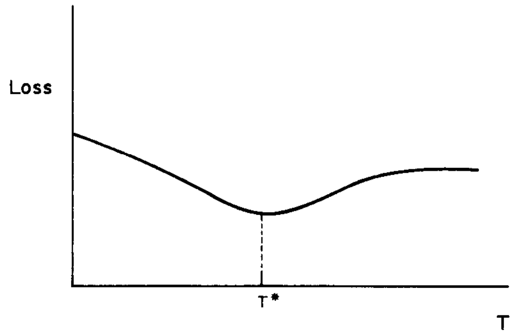

Fig. 2

5. FINAL TIME SELECTION

Certain important considerations are involved in deciding the time horzion, T, and the associated weighting matrix, S. In order to achieve the targets by some predetermined time T, one would place large penalties on deviations at time T through the weighting matrix S. However, this will result in higher instrument settings for the period starting from $t = 0$ to $t = T - 1$. Hence, there will be a trade-off between forcing the targets close to the desired values and the level of instrument settings.† Thus the terminal conditions can be incorporated implicitly in this method of analysis, which could not be done in the previous approaches of Theil and Friedman.

The decision regarding the appropriate time horizon, T, of stabilization policy is a complicated matter. Stabilization actions are taken by policy makers in order to restore the "trajectory" of the endogenous variables, y_t, to their normal paths, $\tilde{y}_t$. When certain endogenous variables deviate from their normal paths beyond acceptable limits, it would invoke a call for a suitable stabilization action. At this time there are various ways to decide upon the time horizon of the policy.

Firstly, it can be a policy decision, dictated by external constraints, as to how rapidly the policy maker requires the economy to be stabilized, which would define the time horizon T. Very often this is the overriding factor.

Secondly, the length of time for which the econometric model is valid can force the decision. This situation arises when the econometric model of the economy has constant parameters. In this case, policy actions are decided for the time duration of the model, at the end of which the model is revised and the situation re-evaluated. On the other hand if an econometric model of the economy with time-varying parameters is available, then decision of T can be taken independently. Our technique is ideally suited for such models, whereas previous approaches could not handle time-varying parameter models.

Thirdly, and most significantly, the policy horizon T inherently depends upon the penalties that the policy maker attributes to deviations of the target variables. By choosing larger Q_t and S matrices, the time period T in which the stabilization is achieved can be shortened to some extent. However, decreasing T beyond that would increase the loss as extraordinary levels of instruments may be required to meet the terminal conditions. Hence for a given Q_t and S matrices, the loss function would have a minimum value for some time horizon T^*, as depicted in Fig. 2.

To obtain the optimal value of time horizon, T^*, the algorithm given in Section 4 could be embedded as a subroutine in a larger algorithm that would iteratively change T as an endogenous variable, until T^* is obtained.‡ This is done by selecting the optimistic (low) value of T and evaluating the optimal value of the loss function. Then T is incremented and the procedure repeated till T^* is obtained.

Finally, it is helpful if the stabilization problem is solved for time $T^* + \tilde{T}$, where $\tilde{T}$ represents the length of the most dominant lag period in the econometric equation. The optimal policy actions

†Certain crucial endogenous variables may be targetted prominently by assigning larger penalties to them, through the S matrix.

‡Such a procedure was suggested by Friedman (1972).

thus obtained are implemented for the time period T^*; this has the effect of reducing swings in the targets that are carried over due to lagged effects in the economy, after policy implementation is stopped at time T^*.

6. STOCHASTIC ECONOMY MODELS

We briefly consider stochastic econometric models in light of our technique. Let

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_r y_{t-r} + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \cdots + \beta_r x_{t-r} + \eta_t,$$

where η_t is a zero mean, white noise process. Then the state-space model can be written as follows:

$$z_{t+1} = Az_t + Bx_t + \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_r \end{bmatrix} \eta_t,$$

$$y_t = Cz_t + \eta_t.$$

In this case we want to minimize the expected value of the loss function

$$E[L] = E\left(\tfrac{1}{2}(y_T - \tilde{y}_T)^T S(y_T - \tilde{y}_T) + \frac{1}{2}\sum_{t=0}^{T-1} [(y_t - \tilde{y}_t)^T Q_t (y_t - \tilde{y}_t) + (x_t - \tilde{x}_t)^T R_t (x_t - \tilde{x}_t)]\right).$$

The solution of this problem will be a combination of the tracking problem discussed in Section 2 and the optimal "reconstruction" problem. In the reconstruction stage, the state variable z_t is reconstructed using a suitable estimator. This is known as the separation principle (see Kailath, 1980; Sage and White, 1977, p. 270) in the optimal control literature; it states that the solution of the stochastic problem is the same as of the deterministic case, except that the state z_t is replaced by its minimum mean square estimator $\hat{z}_t$. Accordingly, equation (14) is modified as follows:

$$\begin{aligned} x_t^* = & -R_t^{-1}B^T P_{t+1}[I - B(R_t + B^T P_{t+1} B)^{-1} B^T P_{t+1}] A\hat{z}_t \\ & + R_t^{-1} B^T P_{t+1}[I - B(R_t + B^T P_{t+1} B)^{-1} B^T P_{t+1}][BR_t^{-1} B^T w_{t+1} - B\tilde{x}_t] - R_t^{-1} B^T w_{t+1} + \tilde{x}_t. \end{aligned} \tag{24}$$

The estimator $\hat{z}_t$ can be obtained using the Kalman–Bucy filter (see Sage and White, 1977, pp. 201–210). The algorithm for finding the optimal instrument setting would essentially remain the same, with an extra step before Step 3 where the Kalman–Bucy filter is implemented.

7. CONCLUSIONS

This paper has attempted to deal conceptually with two drawbacks in the optimal stabilization policy literature, viz. asymmetries in policy loss functions and multi-period time horizons. The piecewise quadratic loss function was embedded in a general state-space model framework to obtain a solution that is feedback in nature. Furthermore, this method gives us the flexibility to incorporate (a) terminal conditions and (b) time-varying loss functions, i.e. the loss matrices Q_t and R_t can be time-varying. The solution procedure is both computationally more efficient and intuitively more plausible as a description of the policy making process. This unified approach can also be used to answer questions related to appropriate time horizons and stochastic models of the economy.

The above framework can also be extended to address some other pertinent questions, such as:
(a) stability of targets and instruments by examining oscillations about their desired paths;
(b) effect of nonlinearities in the feedback loop. Since it is not always possible to implement the control law as given and some nonlinearities do arise during implementation, the robustness of the solution to such effects can also be addressed; and
(c) sensitivity of the feedback solutions to changes in model parameters.
These represent interesting avenues for further research.

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